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ON THE BIFURCATION AND STABILITY OF THE STEADY-STATE MOTIONS OF COMPLEX MECHANICAL SYSTEMS

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Many objects of modern technology (rockets, spaceships, airplanes, gyroscopic devices, centrifuges, etc.) can be modelled in a number of cases by mechanical systems comprised of absolutely rigid bodies and material points and of deformable (liquid and elastic) bodies connected with them. Mechanical systems containing among its parts both subsystems with a finite number of degrees of freedom as well as units with distributed parameters, i.e. continuous media, are called complex systems for brevity. We consider the steady-state motions of complex systems. Stationary values of the potential energy V or of the altered potential energy W of the system correspond to the steady-state motions. The stability problem for the steady-state motions leads to the investigation of the nature of the extremum of the potential energy V or W. The minimum of the potential energy corresponds to a stable motion. In a number of important cases the stability (instability) conditions can be obtained as conditions for the positive definiteness (for sign-alteration together with certain additional conditions) of the second variation $\delta^2 V$ or $\delta^2 W$ of the potential energy. These general results are applied to solving a number of concrete problems on the stability of the steady-state motions of complex systems. Stability conditions for the motion of a rigid body with liquid and elastic parts in various force fields are discussed.

1. We consider complex systems subjected to holonomic constraints. The choice of the model for continuous media occurring as a part in a complex system is essential. To be specific we accept that a liquid body is to be modelled by incompressible homogeneous ideal or viscous Newtonian liquids, while an elastic body, by rigid deformable bodies to be considered as material continua for which the deformation processes are reversible and for which the potential energy of deformation exists [1]. The motion of the complex system is considered relative to a certain inertial coordinate system $O'\xi\eta\zeta$ and it is assumed that under specified external forces it is uniquely determined by specifying the initial conditions and is continuous in time. One of the rigid bodies of the system is taken as the base or supporting body and to it is attached a coordinate system $Ox_1x_2x_3$ with origin at some point O of this body. The position of the system's points in the space $\xi\eta\zeta$ is determined by the generalized Lagrange coordinates q_s ($s = 1, \ldots, n$) of its subsystem with a finite number of degrees of freedom and by the radius-vectors $\mathbf{r}$ (x_1, x_2, x_3) from the origin at point O to the particles of the continuous medium. Then the velocities of the system's points can be represented [2] by certain functions of q_s, $q_s^{\cdot}$, $\mathbf{r}$, $\mathbf{r}^{\cdot}$.

Without citation of the equations of motion of the complex system [2] in explicit form here, we restrict ourselves to an examination of its steady states (equilibria and stationary motions) and of their stability. We assume, further, that the constraints imposed on the system are stationary, while the active forces operating are the position potential forces, the drivatives of the force function depending on the positions of the system's points and, possibly, on some parameter and not depending explicitly on time. Under these conditions there exists the system's potential energy V which, in general case, depends both on the generalized coordinates q_s as well as on the forms of the regions τ_1 and τ_2 occupied at a given instant by the liquid and elastic bodies, so that the potential energy of the system is a function and a functional simultaneously. If the constraints imposed on the system admit of a rotation of the whole system as one rigid body around some fixed straight line $O'\zeta$, and if the forces acting on the system do not yield a moment relative to this line, then there exists an area integral $G_\zeta = k$, where G_ζ is the projection onto the ζ-axis of the system's moment of momentum vector relative to point O' in its absolute motion. In this case we can introduce a functional of the altered potential energy W [2]. Here, among the real motions of the system there can be steady-state motions for which the position coordinates q_s $(s = 1, \ldots, m;\ m \leqslant n)$ and the coordinates x_i of the points of the continuous medium remain constant.

The coordinates q_s and the configuration of the continuous medium, corresponding to the system's steady state when the total heat influx to the continuous medium is zero, are determined, in accordance with the principle of virtual displacements, from the condition

$$\delta F = 0 \tag{1.1}$$

where δ denotes the change in the system's virtual displacement, while the functional F equals V or W. Condition (1.1) is equivalent to the equations

$$\partial F/\partial q_s = 0 \qquad (s = 1, \ldots, m) \tag{1.2}$$

and to functional equations together with natural boundary conditions.

The potential energy of a complex system usually depends on a certain parameter λ which remains constant during any motion of the system. The constant area integral $k = k_0$ serves as such a parameter for the functional W. The system's steady states depend on the values of this parameter and, in general, will vary as it varies continuously, moreover, several states can correspond to one value of the parameter. In the configuration space complemented by a measurement of parameter λ, these states are represented by points of a real "equilibrium" curve consisting, in the general case, of several branches. The separate branches of this curve can intersect each other at bifurcation points. The equilibrium curves yield a global pattern of the distribution of the system's steady states. The determination of the equilibrium positions of the complex system relative to some moving coordinate system O_1xyz is carried out in the same way as for the absolute equilibrium positions under the condition that the energy forces of the transient motion are added to the potential energy. The investigation of the stability of the steady-state motions (equilibria and stationary motions) of complex systems has been successfully carried out by the methods originally developed [3, 4] for rigid and elastic bodies with a liquid filling.

In the papers mentioned two definitions of stability are proposed. According to one of them, by the stable motion of a complex system we can understand Liapunov stability with respect to the variables q_s ($s = 1,\ldots, k$; $k = n$ in the case of an equilibrium and $k = m < n$ in the case of stationary motions), q_i ($i = 1,\ldots, n$) and p_j ($j = 1,\ldots, r$), where the quantities p_j are certain integral characteristics of the motion of the continuous medium and there are a finite number r of them [2]. The actual choice of the quantities p_j depends on the problem being considered and should take into account the physical conditions. In the statement indicated the problem of the stability of the motion of a system with an infinite number of degrees of freedom reduces to the investigation of the stability with respect to a finite number $k + n + r$ of quantities q_s, q_i, p_j, i.e. is posed as a problem of stability with respect to a part of the variables (*). Under such a posing of the problem, Liapunov's second method proves to be effective, however, not in its standard form but in a somewhat modified one. The thing is that when investigating stability with respect to not all the variables but to a part of them the classical theorems of Liapunov's second method are not applicable directly since we cannot, as a rule, succeed in constructing Liapunov functions which depend only on the variables of interest to us. The application of the method to such problems is based on certain theorems on stability with respect to a part of the variables (see [3], Chapter 3, Section 2), being modifications of Liapunov's theorems. The expressions playing the role of Liapunov functions in the case of complex systems are functionals of the initial variables. Chetaev's method [6] is effective for constructing them. The application of modifications of the theorems of Lagrange and Routh and of their generalizations is effective in the case of the steady-state motions. The given statement of the stability problem has turned out to be fruitful also in problems of stability of motion of a continuous medium under a suitable choice of the integral characteristics of the medium's motion. The other definition of the stability of the steady-state motions of complex systems is a synthesis of definitions of Liapunov stability with respect to the variables q_s, $q_i^{\cdot}$ and to the equilibrium forms of the units with distributed parameters. The characteristics of the deviation of the perturbed form from the unperturbed one can be introduced differently by taking as such, as Liapunov had suggested, the "deviation" or else some other quantities, for example, the L_2-norms of the relative displacements $\| \mathbf{u} \|$ [4]. We remark that stability with respect to the equilibrium form of the continuous medium refers, essentially, also to the class of problems on stability with respect to a part of the variables.

According to the theorems proved for rigid and elastic bodies with a liquid filling and easily extendible to complex systems, the equilibrium (stationary motion) is stable if for it the functional F has a minimum $F^{(0)}$ and is unstable if for it F does not have a minimum and can take negative values in an arbitrarily small neighborhood of the steady-state motion, moreover, the signs of the expressions $F - F^{(0)} = F^{(2)} + F^{(3)} + \ldots$ and $2F^{(2)} + 3F^{(3)} + \ldots$ are determined by the second-order terms $F^{(2)}$ in the Taylor series expansion of the functional (and here gyroscopic stabilization is impossible). These results are intensified in the case when the dissipation occurs on every real motion of the system other than the steady-state motions [3, 4]. We stress

*) The opinion in [5] to the effect that the system being considered is by the same token changed into a finite-dimensional system is erroneous; as a matter of fact it remains as it is and is described by the original differential equations of motion.

that the presence (absence) of a minimum for F is to be interpreted as the positive definiteness (sign alteration) of the functional $F - F^{(0)}$.

Under a continuous variation of the constant parameter λ on which F depends, the steady states and the functional F vary continuously, and for all points of the equilibrium curve for which F retains a minimum, the steady states remain stable. A change in stability on a specified branch of the equilibrium curve can take place only at a bifurcation point where the equation $\delta^2 F = 2F^{(2)} = 0$ has a nontrivial solution. The law for changes in stability for a fixed value of the parameter established by Poincaré, also holds. We remark that here we have in mind an essentially constant parameter which remains constant on any motion of the system [7].

In ordinary cases the question of the nature of the extremum of functional F is settled by investigating its second variation $\delta^2 F$ whose form depends not only on the functional F itself but also on the choice of the functions characterizing the deviation of the perturbed form of the continuous medium from the unperturbed one and satisfying specified conditions. If the second variation $\delta^2 F$ is positive definite, then F has a minimum, however, if $\delta^2 F$ can take negative values, then F does not have a minimum. The question of the nature of the extremum of functional F is settled by terms of higher than second order only in the singular cases when $\delta^2 F$ is nonnegative; these cases are not considered further.

Without loss of generality we assume that for the steady-state motion being examined the generalized coordinates $q_s = 0$, so that in its neighborhood $\delta q_s = q_s$. For complex systems the second variation of functional F consists of three parts

$$\delta^2 F = F_1(q) + F_2(n) + F_3(q, n)$$

Here $F_1(q)$ is a quadratic form in the generalized coordinates q_s, coinciding with $\delta^2 F$ for a "hardened" system obtained from the original one by a hardening of the continuous media; $F_2(n)$ is a quadratic functional reflecting the change in the forms of the corresponding units of the system with distributed parameters, described by the vector-valued function n; $F_3(q, n)$ is a functional bilinear in the function n and in the coordinates q_s, characterizing the mutual influence of a change in position of a subsystem with a finite number of degrees of freedom and of the deformation of its units with distributed parameters.

We mention two methods for establishing the conditions for the positive definiteness of $\delta^2 F$. One of them, developed in [8–10], consists of the following. Suppose that the functional $F_2(n)$ is positive definite. With the aid of the solution $n^*(q)$ of the equation

$$\delta\,[F_2(n) + F_3(q, n)]_{q=\mathrm{const}} = 0 \tag{1.3}$$

the second variation $\delta^2 F$ can be reduced to the form

$$\delta^2 F = F_1(q) + F_2(n - n^*) + {}^1/_2 F_3(q, n^*(q)) \tag{1.4}$$

where $F_3(q, n^*(q))$ is a quadratic form in the coordinates q_s since the solution $n^*(q)$ of Eq. (1.3) is a linear function of q_s. Thus, $\delta^2 F$ represented as a sum of two independent parts and the conditions for its positive definiteness, consists of the conditions for the positive definiteness of functional F_2 and of the quadratic form

$$U = F_1(q) + {}^1/_2 F_3(q, n^*(q)) \tag{1.5}$$

The fulfillment of the conditions for the positive definiteness of functional F_2 ensures the stability of the "equilibrium" of the units with distributed parameters for $q_s = 0$, while the conditions for the positive definiteness of quadratic form U can be interpreted as the conditions for the stability of an "equivalent" mechanical system consisting of rigid bodies and material points and having, generally speaking, a configuration other than that of a hardened system.

In another method [11] for establishing the conditions for the positive definiteness of the second variation it is assumed that the quadratic form $F_1(q)$ is positive definite. With the aid of the solution $q^*(n)$ of the equation

$$\delta [F_1(q) + F_3(q, n)]_{n=\text{const}} = 0$$

the second variation can be reduced to the form

$$\delta^2 F = F_1(q - q^*(n)) + F_2(n) + {}^1/{}_2 F_3(q^*(n), n) \qquad (1.6)$$

where $F_3(q^*(n), n)$ is a quadratic form in the functional n. The conditions for the positive definiteness of (1.6) are made up from the conditions for the positive definiteness of the quadratic form F_1 and of the quadratic functional

$$\Phi(n) = F_2(n) + {}^1/{}_2 F_3(q^*(n), n)$$

The positive definiteness of F_1 guarantees the stability of the subsystem with a finite number of degrees of freedom with units with distributed parameters, hardened in the steady state, while the positive definiteness of functional $\Phi(n)$ guarantees the stability of a certain equivalent system consisting of units with distributed parameters and differing generally speaking, from the original subsystem with distributed parameters.

Both methods yield necessary and sufficient conditions for the positive definiteness of $\delta^2 F$. In these methods, by means of estimates of a different sort, we can obtain sufficient conditions for the positive definiteness of $\delta^2 F$, for example, those derived in [11–13]. It is not difficult to show that $F_3(q, n^*(q)) \leqslant 0$ and $F_3(q^*(n), n) \leqslant 0$. Therefore, the stability conditions for the equilibrium of the equivalent system are worse than the analogous conditions for the hardened system or for the subsystem with distributed parameters.

In this becomes apparent a common property of systems with deformable elements, being that deformability renders a destabilizing influence on the system's equilibrium in comparison with the same configuration consisting of undeformable elements. For example, this property was repeatedly noted for rigid bodies with cavities containing a liquid. In certain practical problems the possibility occurs of changing the system's mass distribution, for example, by spreading out its individual parts by considerable distances. These changes promote the stabilization of the equilibrium of a system of rigid bodies, however, they usually intensify the deformability of the system's elements, which can lead not only to a significant lowering of the expected effect of stabilization, but, in isolated cases, also to a destabilization of the system's equilibrium.

The problem of the positive definiteness conditions for functionals $F_2(n)$ and $\Phi(n)$ can be reduced to the problem of the positiveness of the smallest eigenvalue of the corresponding boundary value problem. Numerical methods can be used to determine the coefficients of the quadratic form $F_3(q, n^*(q))$. Later on we consider a number of problems of the stability of the steady-state motions of concrete complex systems, being of specific independent interest.

2. Let us examine the motion around a fixed point O of a rigid body having a cavity partially filled with a liquid of density ρ, neglecting its surface tension. To the rigid body there is fixedly connected the axis of symmetry of a dynamically balanced rotor whose constant relative gyrostatic moment R is directed along the x_3-axis (with unit vector $\mathbf{i}_3$) ; other rigid bodies and material points also are connected, their generalized relative coordinates are q_s $(s = 1, \ldots, n)$. We assume that gravity forces as well as internal forces with potential energy $V(q_1, \ldots, q_n)$ act on the system. The altered potential energy of the system is

$$W = \frac{1}{2J}(k_0 - R\mathbf{i}_3\gamma)^2 + V(q_1, \ldots, q_n) + Mg(x_{c1}\gamma_1 + x_{c2}\gamma_2 + x_{c3}\gamma_3)$$

$$J = A\gamma_1^2 + \beta\gamma_2^2 + C\gamma_3^2 - 2D\gamma_2\gamma_3 - 2E\gamma_3\gamma_1 - 2F\gamma_1\gamma_2$$

Here $\boldsymbol{\gamma}(\gamma_1, \gamma_2, \gamma_3)$ is the unit vector along the axis $O\zeta$ directly vertically upwards, J is the system's moment of inertia relative to the ζ-axis; A, B, C, D, E, F are the moments and the products of inertia, x_{ci} are the coordinates of the system's center of mass. Equations of form (1.2) admit of the solution

$$\gamma_1 = \gamma_2 = 0, \; \gamma_3 = 1, \; q_i = q_i^\circ$$

if conditions

$$x_{c1} = x_{c2} = 0, \; D = E = 0$$

are fulfilled for an arbitrary constant quantity $\omega = (k_0 - R)/J_0$, i.e. if the system's center of gravity is located on the x_3-axis which is the principal inertia axis of the system.

From a condition of form (1.1) we obtain the equation of the liquid's free surface

$$^1/_2\omega^2(x_1^2 + x_2^2) - gx_3 = c \tag{2.1}$$

In the given case the quadratic form (1.6) is the following:

$$U = [(C^\circ - A^\circ)\omega^2 + R\omega - Mgx_{c3}^\circ - a]\gamma_1^2 + [(C^\circ - B^\circ)\omega^2 + R\omega -$$

$$Mgx_{c3}^\circ - a]\gamma_2^2 + 2\sum_{i=1}^{n}\left[\left(\frac{\partial^2 W}{\partial\gamma_1\partial q_i}\right)^\circ\gamma_1 + \left(\frac{\partial^2 W}{\partial\gamma_2\partial q_i}\right)^\circ\gamma_2\right](q_i - q_i^\circ) +$$

$$\sum_{i,j=1}^{n}\left(\frac{\partial^2 W}{\partial q_i\partial q_j}\right)^\circ(q_i - q_i^\circ)(q_j - q_j^\circ) \tag{2.2}$$

Here, in the case of the intersection of surface (2.1) with the cavity's walls along circles of radii R_1, R_2 $(R_1 > R_2 \geqslant 0)$ and with centers on the x_3-axis,

$$a = \pi g\rho\int_{R_1}^{R_2}\left[\frac{\omega^2}{g^2}\left(\frac{\omega^2}{2}r^2 - c\right) + 1\right]r^3 dr$$

For expression (2.2) to be positive definiteness it is necessary and sufficient to fulfil the Sylvester conditions

$$(C^\circ - A^\circ)\omega^2 + R\omega - Mgx_{c3}^\circ - a > 0 \quad (A^\circ \geqslant B^\circ) \tag{2.3}$$

$$\Delta_{2+i} > 0 \qquad (i = 1, \ldots, n)$$

where Δ_{2+i} are the principal diagonal minors of the discriminant of quadratic form (2.2), corresponding to the variables q_i $(i = 1, \ldots, n)$. Inequalities (2.3) are sufficient conditions for the stability of the motion being considered.

In the case of a viscous liquid and of the action of dissipative forces Q_i such that $Q_1q_1^{\cdot} + \ldots + Q_nq_n^{\cdot} \leqslant 0$, where the equality sign holds only for $q_i^{\cdot} = 0$ $(i = 1, \ldots, n)$, when conditions (2.3) are fulfilled the system's perturbed motion, sufficiently close to the unperturbed one, tends asymptotically to a uniform rotation around the vertical of the whole system, except the rotor, as one rigid body. When one or several of inequalities (2.3) change to the opposite sign, the unperturbed motion becomes unstable. The quantity a occurring in condition (2.3) is of order ω^2 as $\omega \to \infty$. Although the integrand contains ω to the fourth degree, $R_2 \to R_1$ as $\omega \to \infty$ for any bounded cavity.

The limit case $\omega = \infty$ is of definite practical interest and permits us to describe comparatively simply the evolution of the stability conditions as the amount of liquid in the cavity varies. In this case the equation of the free surface (2.1) has the form

$$x_1^2 + x_2^2 = R_1^2 = R_2^2 = b^2 \qquad (2.4)$$

If the cavity is bounded by the planes $x_3 = h \pm d$, then the first of inequalities (2.3) can be written in the following manner:

$$C^\circ - A^\circ - 2\pi\rho b^2 d\,\frac{3h^2 + d^2}{3} > 0$$

or

$$C_1 - A_1 - {}^1/_2\pi\rho d b^4 > 0$$

Here C_1, A_1 are the moments of inertia of a fictitious rigid body obtained from the original hardened system by filling cylinder (2.4) formed by the liquid's free surface The stability condition for such a rigid body has the form

$$C_1 - A_1 > 0$$

and coincides with the stability condition for a rigid body with a cavity wholly filled with a liquid. Thus, in the end, the lessening of the amount of liquid in the cavity of a rapidly rotating body worsens the stability, i.e. proves to have a destabilizing influence, although the difference between the axial and the transverse moments of inertia of the system is increased here (as is easy to see, $C_1 - A_1 < C^\circ - A^\circ$ for small r).

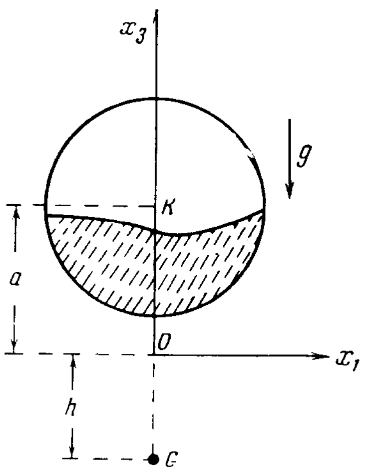

Fig. 1

3. The study of cavities of concrete form allows us to construct a complete pattern of the distribution of the equilibrium positions of the complex system and of their evolution and bifurcation as the system parameters vary. As an elementary example we consider the equilibrium of a heavy physical pendulum with a spherical cavity partially filled with a liquid and with a horizontal axis of suspension (Fig. 1). The most interesting case is when the body's center of gravity G and the cavity's center K lie in one plane with the swing axis O and are located on different sides of it (Fig. 1). In this case, as is not difficult to establish, the system has two equilibrium positions: $q = 0$ and $q = \pi$ for any amount of liquid in the cavity. Here q is the

angle between the x_3-axis and the vertical. Moreover, any position of the system can be considered an equilibrium position if $Mh = ma$. Here M, m are the mass of the body and the mass of the liquid, respectively, while the quantities a and h are geometric characteristics (Fig. 1).

The pattern of the distribution of the equilibrium positions with the demarkation of the stable positions and their evolution under a change in the amount of liquid is shown in Fig. 2. For a small amount $(m < m_1 = Mh/a)$ of liquid in the body's cavity the equilibrium position $q = 0$, at which the body's center of gravity G is located below the swing axis O, is stable. For a sufficiently large amount $(m > m_1)$ of liquid the other equilibrium position $q = \pi$, at which the cavity's center K is located below axis O, is stable. The change in stability on the branches $q = 0$, $q = \pi$ takes place at the bifurcation points when $m = m_1$.

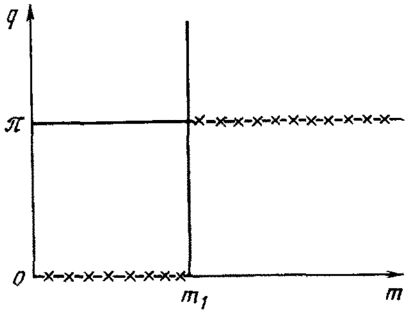

Fig. 2

The study of more complex cavities leads to more complicated bifurcation patterns. For example, a cavity in the form of a rectangular parallelepiped was analyzed in [12]. The investigation of special cases in [12] has allowed us also to establish that the form of the cavity essentially determines the nature of the influence of the liquid's surface tension, which was not taken into account in the examples presented, on the stability condition.

4. Let us consider the problem of the stability of the uniform vertical rotation around a fixed point of a rigid body with a thin rectilinear nonextendible elastic rod rigidly fixed to it, in a uniform gravity force field. We introduce two rectangular coordinate systems with origin at a fixed point O of the body: an inertial one $O\xi\eta\zeta$, whose ζ-axis we direct vertically upward, and a moving system $Ox_1x_2x_3$, whose axes we direct along the principal inertia axes of the body for the point O. Let $\mathbf{i}_j$ be the unit vectors directed along the x_j-axes $(j = 1, 2, 3)$. We denote the unit vector of the ζ-axis by $\boldsymbol{\gamma}$ and its projection onto the x_j-axes by γ_j. We assume that one end of a rod of length l is fastened to the body at a distance a from point O and, in its undeformed state, is directed along the x_3-axis. The x_3x_1- and x_2x_3-planes serve as its planes of symmetry. By

$$\mathbf{u}(t, s) = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3, \quad 0 \leqslant s \leqslant l, \quad t \geq t_0$$

we denote the vector of elastic displacement of the points of the rod's axis. The condition that the rod is nonextendible leads to the relation

$$u_3' = -\frac{1}{2}(u_1'^2 + u_2'^2) \qquad (u' = \partial u/\partial s)$$

while the condition that the rod's end is fastened to the body leads to the boundary condition

$$u_1 = u_2 = 0, \quad u_1' = u_2' = 0 \quad \text{for } s = 0, \quad t \geq t_0$$

The system's altered potential energy is

$$W = \frac{k_0^2}{2J} + \Pi$$

$$J = J_1\gamma_1^2 + J_2\gamma_2^2 + J_3\gamma_3^2 + \sigma\rho\int_0^l \Big\{u_2^2\gamma_1^2 + u_1^2\gamma_2^2 + (u_1^2 + u_2^2)\gamma_3^2 - \Big[a(l-s) + \frac{1}{2}(l^2 - s^2)\Big](u_1'^2 + u_2'^2)(\gamma_1^2 + \gamma_2^2) - 2(a+s)(u_1\gamma_1 + u_2\gamma_2)\gamma_3 - 2u_1u_2\gamma_1\gamma_2\Big\}ds$$

$$\Pi = Mg(x_{10}\gamma_1 + x_{20}\gamma_2 + x_{30}\gamma_3) + g\sigma\rho\int_0^l \Big[\gamma_1 u_1 + \gamma_2 u_2 - \frac{1}{2}\gamma_3(l-s)(u_1'^2 + u_2'^2)\Big]ds + \frac{1}{2}E\int_0^l (I_2 u_1''^2 + I_1 u_2''^2)\,ds$$

Here J is the system's moment of inertia relative to the ζ-axis, Π is the potential energy of the gravity forces and of the elastic deformation of the rod, J_i are the undeformed system's moments of inertia relative to the x_i-axes, σ is the area of the rod's cross section, ρ is its density, M is the mass of the system, g is the gravitational acceleration, x_{i0} are the coordinates of the center of gravity of the system in its undeformed state, E is the Young's modulus, EI_1 and EI_2 are the flexural rigidities.

When $x_{10} = x_{20} = 0$, the equations of stationary motions of form (1.2) admit of a solution describing the uniform rotation of the rigid body with the rod undeformed around the vertical with an angular velocity ω,

$$\gamma_1 = \gamma_2 = 0, \ \gamma_3 = 1, \ u_1 = u_2 = 0 \tag{4.1}$$

In the neighborhood of the unperturbed motion (4.1) we represent the expression for $\delta^2 W$ in the form (1.6)

$$\delta^2 W = [(J_3 - J_1)\omega^2 - Mgx_{30}]\Big\{\gamma_1 + [(J_3 - J_1)\omega^2 - Mgx_{30}]^{-1}\sigma\rho \times \int_0^l [g + \omega^2(a+s)]\,u_1 ds\Big\}^2 + [(J_3 - J_2)\omega^2 - Mgx_{30}]\Big\{\gamma_2 + [(J_3 - J_2)\omega^2 - Mgx_{30}]^{-1}\sigma\rho\int_0^l [g + \omega^2(a+s)]\,u_2 ds\Big\}^2 + V \tag{4.2}$$

Here

$$V(u_1, u_2) = \sigma\rho\int_0^l \{E_*(I_2 u_1''^2 + I_1 u_2''^2) - g(l-s)(u_1'^2 + u_2'^2) - \omega^2(u_1^2 + u_2^2)\} \times ds - [(J_3 - J_1)\omega^2 - Mgx_{30}]\Big\{[(J_3 - J_1)\omega^2 - Mgx_{30}]^{-1}\sigma\rho\int_0^l [g + \omega^2(a+s)] \times u_1 ds\Big\}^2 - [(J_3 - J_2)\omega^2 - Mgx_{30}]\Big\{[(J_3 - J_2)\omega^2 - Mgx_{30}]^{-1}\sigma\rho \times \int_0^l [g + \omega^2(a+s)]\,u_2 ds\Big\}^2 \qquad (E = \delta\rho E_*) \tag{4.3}$$

Let the conditions
$$(J_3 - J_i)\,\omega^2 - Mgx_{30} > 0 \qquad (i = 1, 2) \tag{4.4}$$
be satisfied, being sufficient stability conditions for the uniform vertical rotation (4.1) of the heavy rigid body with the rod undeformed ($u_1 = u_2 \equiv 0$). Then, using inequalities of the form
$$\left\{\sigma\rho \int_0^l [g + \omega^2 (a + s)]\, u\,ds\right\}^2 \leqslant h\sigma\rho \int_0^l u^2 ds$$
where
$$h = \sigma\rho \int_0^l [g + \omega^2 (a + s)]^2\, ds = J_* \omega^4 + 2mgx_{3*}\omega^2 + mg^2$$
$m = \sigma l\rho$ is the rod's mass, J_* is the undeformed rod's moment of inertia relative to point O, and x_{3*} is the coordinate along the x_3-axis of the undeformed rod's center of mass, from (4.3) we obtain the inequality
$$V(u_1, u_2) \geq \sigma\rho \int_0^l \{\sigma (\lambda_1 u_1'^2 + \lambda_2 u_2'^2) + \{\lambda_1 - \omega^2 - h\,[(J_3 - J_1)\,\omega^2 - Mgx_{30}]^{-1}\} \times$$
$$u_1^2 + \{\lambda_2 - \omega^2 - h\,[(J_3 - J_2)\,\omega^2 - Mgx_{30}]^{-1}\}\, u_2^2\}\, ds \tag{4.5}$$

Here λ_1 and λ_2 are the minima of the functionals
$$\Phi_1(u) = \left\{\int_0^l (u^2 + \sigma u'^2)\, ds\right\}^{-1} \int_0^l \{E_* I_2 u''^2 - g\,(l - s)\, u'^2\}\, ds$$
$$\Phi_2(u) = \left\{\int_0^l (u^2 + \sigma u'^2)\, ds\right\}^{-1} \int_0^l \{E_* I_1 u''^2 - g\,(l - s)\, u'^2\}\, ds$$

in the class of functions $u(s)$, $0 \leqslant s \leqslant l$, continuously differentiable up to fourth order, satisfying the conditions $u(0) = 0$, $u'(0) = 0$ From (4.2), (4.4) and (4.5) it follows that the inequalities
$$\lambda_i > 0, \quad \lambda_i - \omega^2 > h\,[(J_3 - J_i)\,\omega^2 - Mgx_{30}]^{-1} > 0 \quad (i = 1, 2)$$
serve as the sufficient conditions for the positive definiteness of $\delta^2 W$ and, consequently, are the sufficient stability conditions for the unperturbed motion (4.1).

5. Let us consider the motion in a central Newtonian force field of a rigid body supporting on itself thin or thin-walled nonextendible elastic rods each of which has two planes of symmetry. We assume that three pairs of elastic rods of length l are fastened to the body at like distances a from the center of mass O of the rigid body and, in the undeformed state, are positioned along the principal central inertia axes of the body, moreover, the principal inertia planes serve as the symmetry planes of the rods.

The problem of the stability of the relative equilibrium of such a system on a circular orbit was investigated in [11] by means of representing the second variation $\delta^2 W$ in the form (1.6). This problem is solved below by means of representing $\delta^2 W$ in the form (1.4). Then for the quadratic form (1.5) we have
$$U = {}^1/_2\Omega^2\,[(J_2 - J_1 - b_1)\,\beta_1^2 + 3\,(J_1 - J_3 - b_2)\,\gamma_1^2 +$$
$$4\,(J_2 - J_3 - b_3)\,\gamma_2^2] \tag{5.1}$$

where the quantities $b_i > 0$ are computed from the solutions u_{ij}^* of equations of type (1.3). Here J_i are the principal central moments of inertia of the rigid body with the rods undeformed, $m = \sigma l\rho$ is the mass of a rod, σ is a rod's cross-sectional area, ρ is the density of the rods, Ω is the angular velocity of the motion along the orbit of the system's center of mass.

The conditions for positive definiteness of (5.1) have the form

$$J_2 - J_1 - b_1 > 0,\ J_1 - J_3 - b_2 > 0,\ J_2 - J_3 - b_3 > 0$$

Numerical methods can be used to compute the constants b_i. Sufficient stability conditions can be obtained also without the use of numerical methods if we make estimates of the functionals occurring in $\delta^2 V$. We can then show that the conditions

$$3k_1^4 < \nu_*^4,\quad 1 - {}^1/_4\, k_2^4 > 0,\quad 3k_2^4 < \nu_*^4 (1 - {}^1/_4\, k_2^4) \quad (k_i^4 = \rho\sigma\Omega^2 l^4 (EI_i)^{-1})$$

serve as the sufficient conditions for the positive definiteness of functional F_2. Here EI_i are the flexural rigidities of the rods $(i = 1, 2, 3)$, $\nu_* = 1.875$ is the first root of the equation $1 + \operatorname{ch}\nu \cos\nu = 0$; for simplicity of computation we assume that $a = 0$. The quantities b_i can be replaced by

$$b_1^\circ = 2ml^2\left[\frac{11}{420}\,\frac{k_2^4}{1 - {}^1/_4 k_2^4} + g_1(\nu_1)\right],\quad b_2^\circ = 2ml^2\left[\frac{11k_3^4}{140 + 33k_3^4} + g_2(\nu_2)\right]$$

$$b_3^\circ = 2ml^2\left[\frac{11k_3^4}{3(11k_3^4 + 35)} + \frac{4}{3}\, g_2(\nu_4)\right]$$

where $b_i \leqslant b_i^\circ\ (i = 1, 2, 3)$. Here

$$g_1(\nu) = \frac{1}{3} - \frac{\operatorname{sh} 2\nu - \sin 2\nu}{2\nu^3(2 + \operatorname{ch} 2\nu + \cos 2\nu)},\qquad g_2(\nu) = \frac{\operatorname{ch}\nu \sin\nu - \operatorname{sh}\nu \cos\nu}{\nu^3(1 + \operatorname{ch}\nu\cos\nu)} - \frac{1}{3}$$

$$\nu_1^4 = {}^1/_4\, k_1^4,\qquad \nu_2^4 = 3k_1^4,\qquad \nu_*^4 = 3k_2^4 / (1 - {}^1/_4\, k_2^4)$$

Let us consider in more detail the sufficient stability conditions for a rigid body with one pair of elastic rods having, in the relative equilibrium position, a direction tangent to the orbit. The sufficient stability conditions have the form

$$3k_1^4 < \nu_*^4,\ A_2 > A_3,\ A_2 - A_1 + {}^2/_3\, ml^2[1 - 3g_1(\nu_1)] > 0$$

$$A_1 - A_3 - {}^2/_3\, ml^2\,[1 + 3g_2(\nu_2)] > 0$$

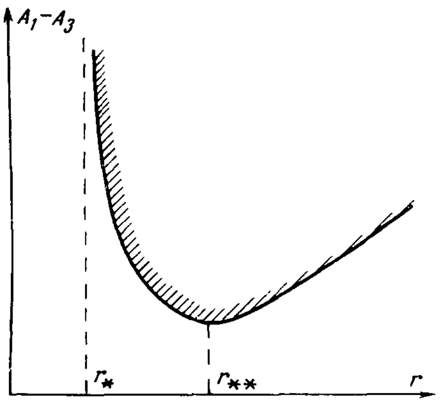

Fig. 3

Here A_i are the principal central moments of inertia of the rigid body. In particular, for circular rods with cross-sectional radius r the stability region in the plane of the parameters $A_1 - A_3$ and r is indicated on Fig. 3. The value r_* corresponds to the value $\nu_2 = \nu_*$, i.e. to that value of the parameter for which the loss of stability of the rectilinear form of the rod takes place. Calculations show that when all the remaining paramaters of the rod are fixed and for a fixed angular velocity Ω of orbital motion there exists an optimal rod radius r_{**} for which the stability region is the largest. For $r > r_{**}$ the stability region contracts because of the increase of rod mass $(m = \rho\pi r^2 l)$, while for $r < r_{**}$ it contracts because of the increase of rod deformability since $\nu \to \nu_*$ and $g(\nu) \to \infty$ as $r \to r_*$.

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